3.9 Inverse Trigonometric Functions:

Each of the 6 trig functions is periodic and so not one-to-one on its whole domain. However by restricting the domain it is possible to make each trig function) one-to-one.

$\sin^{-1}$: Where should we restrict sin to get an one-to-one function?

Graph:

Therefore $\sin^{-1}$ is defined on $[-\pi/2, \pi/2]$. Its graph is above.

Properties:

1. $\sin^{-1}(\sin\theta) = \theta$ provided $-\pi/2 \le \theta \le \pi/2$
2. $\sin(\sin^{-1}(x)) = x$,
3. $\sin^{-1}(x)$ is defined for $-\pi/2 \le x \le \pi/2$.
4. $\dfrac{d}{dx}\sin^{-1}(x) =$ ______________

Verify of Property 4: First of all $\sin^{-1}(x)$ is differentiable except at those points corresponding to where the derivative of $\sin\theta$ is 0: $\cos\theta = 0$. Since $-\pi/2 \le \theta \le \pi/2$ $\theta = \pm\pi/2$ corresponding to $x = \pm 1$. So $\sin^{-1}(x)$ is differentiable $-1 < x < 1$ (not $x = \pm 1$). Compute it: By the chain rule

$$\begin{aligned}\frac{d}{dx}\sin(\sin^{-1}(x)) &= \frac{d}{dx}x \\ \cos(\sin^{-1}(x))\frac{d}{dx}\sin^{-1}(x) &= 1\end{aligned}$$

For $-\pi/2 \le \theta\pi/2$, $\cos\theta = \sqrt{1-\sin^2\theta}$ so that $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$-1 < x < 1$.

Example: Differentiate $\sin^{-1}(x/2)$.

$\tan^{-1}(x)$**:** Restrict $\tan\theta$ to $-\pi/2 < \theta < \pi$.

Graph:

Therefore $\tan^{-1}$ is defined on $(-\pi/2, \pi/2)$. Its graph is above.
Properties:

1. $\tan^{-1}(\tan\theta) = \theta$ provided $-\pi/2 < \theta < \pi/2$
2. $\tan(\tan^{-1}(x)) = x$,
3. $\tan^{-1}(x)$ is defined for all x.
4. $\frac{d}{dx}\tan^{-1}(x) =$ __________

Verify of Property 4: First of all $\tan^{-1}(x)$ is differentiable except at those points corresponding to where the derivative of $\tan\theta$ is 0: $(\sec\theta)^2 = 0$. There are no such θ so that $\tan^{-1}(x)$ is differentiable for all x (not $x = \pm 1$). Compute it: By the chain rule

$$\begin{aligned} \frac{d}{dx}\tan(\tan^{-1}(x)) &= \frac{d}{dx}x \\ (\sec(\tan^{-1}(x)))^2 \frac{d}{dx}\tan^{-1}(x) &= 1 \\ 1 + (\tan(\tan^{-1}(x)))^2 \frac{d}{dx}\tan^{-1}(x) &= 1 \end{aligned}$$

because $(sec\theta)^2 = 1 + (\tan\theta)^2$. Therefore

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

Example: Differentiate $y = \arctan(x/3)$
$\cos^{-1}(x)$**:** Restrict $\cos\theta$ to $0 \le \theta \le \pi$ so that it is one-to-one.
Graph:

Therefore $\cos^{-1}$ is defined on $[0, \pi]$. Its graph is above.
Properties:

1. $\cos^{-1}(\cos\theta) = \theta$ provided $0 \le \theta \le \pi$
2. $\cos(\cos^{-1}(x)) = x$,

3. $\cos^{-1}(x)$ is defined only for $-1 \le x \le 1$.

4. $\dfrac{d}{dx}\cos^{-1}(x) = -\dfrac{1}{\sqrt{1-x^2}}$

$\sec^{-1}(x)$

Properties:

1. $\sec^{-1}(\sec\theta) = \theta$ provided $0 \le \theta \le \pi$, $\theta \ne \pi/2$

2. $\sec(\sec^{-1}(x)) = x$, for $|x| > 1$.

3. $\cos^{-1}(x)$ is defined only for $|x| > 1$.

4. $\dfrac{d}{dx}\sec^{-1}(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$

Property 4 follows by differentiating $\sec(\sec^{-1} x) = x$

$$\frac{d}{dx}\sec(\sec^{-1} x) = \frac{dx}{dx} = 1,\ \ \sec(\sec^{-1} x)\tan(\sec^{-1}(x)) = 1$$

and of course $\sec(\sec^{-1}(x)) = x$ whereas $\tan(\sec^{-1}(x))$ we can evaluate from the identity $(\tan\theta)^2 = \sec(\theta)^2 - 1$ and so if $\theta = \sec^{-1}(x)$ $\tan\sec^{-1} x = x^2 - 1$ and of course $\tan\theta > 0$ for $0 \le \theta < \pi/2$ and $\tan\theta < 0$ for $\pi/2 < \theta \le \pi$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$